Taylor Series : Level - I
Essence and Multivariate Expansion

Dr Anand M Shivapuji
Chief Architect- www.thermofluidscitech.com

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Contents

1 Pre-requisites

1.1 General

Elementary knowledge about algebra, factorials and calculus

1.2 Infinite Series

In very simple terms, an infinite series is basically a sum of infinite sequential terms. A very simple example would be;

$a_0 + a_1 + a_2 + a_3 + ... + a_n + ...$

In strict mathematical terms, there is one more constraint in that the sequential terms must have some kind of well defined relation. A few examples would be;

$x + x^2 + x^3 + x^4 + ... + x^n + ...$

$1/x + 1/x^2 + 1/x^3 + 1/x^4 + ... + 1/x^n + ...$

$$f(x) + \frac{d}{dx}f(x) + \frac{d^2}{dx^2}f(x) + \frac{d^3}{dx^3}f(x) + ... + \frac{d^n}{dx^n}f(x) + ...$$

2 Introduction to Taylor's Series

2.1 Preliminaries

Consider a simple experiment involving the measurement of temperature of a body or some location. Let the temperature be measured with time i.e., the temperature be noted every second, minute, hour as the requirement may be. Some potential examples are;

(a) Measurement of temperature of water in a kettle kept on a heater

(b) Measurement of temperature of a hot cup of tea/coffee kept in the open

(c) Measurement of temperature of a location over a 24 hour period (ambient temperature measurement)

It is evident that, in each of the above cases, the temperature varies as a function of time. For the kettle of water, the temperature continues to rise till it reaches the boiling point while the hot cup of tea/coffee, the temperature continues to reduce till it reaches room temperature. The temperature at a location follows kind of a wave pattern; gradually increasing as the day progresses, reaching a peak at noon and then continuing to decrease. Some hypothetical examples corresponding to the described cases are presented in Figure 1 (a), (b) and (c).

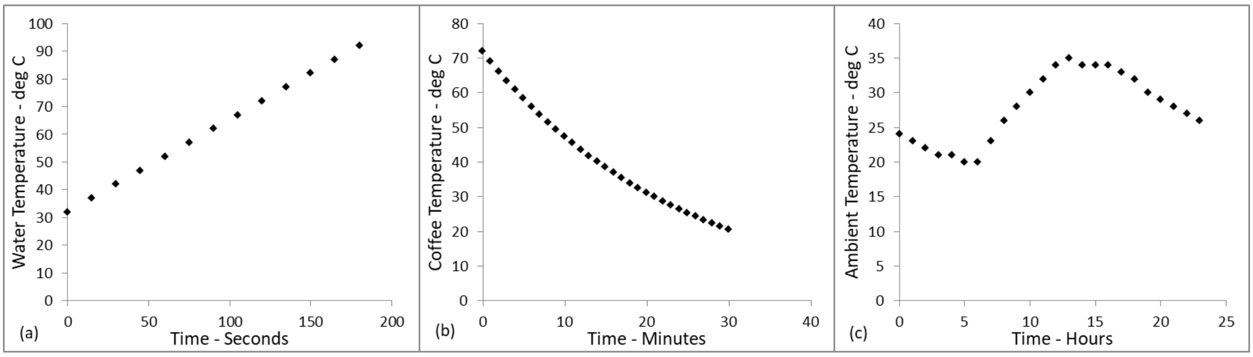

Figure 1: Temperature measurement examples

Based on the available experimental data, a function relationship can be established between temperature (dependent parameter) and time (independent parameter) using the curve fit approach. The functional relation can be represented as $T = f(t)$. For the experimental data presented in Figure 1 (a), (b) and (c), the curve fit based polynomial functional relations can be of the type as below;

(a) $T = 0.3333t + 32$

(b) $T = 0.0344t^2 - 2.7065t + 71.3690$

(c) $T = -0.00001t^6 + 0.0009t^5 - 0.0233t^4 + 0.2667t^3 - 1.0472t^2 + 0.2637t + 23.8840$

In the above set of equations, temperature T has the units deg C while time t has units of seconds, minutes and hours for the equations set (a), (b) and (c) respectively. The curve fit functions can now be used to estimate the temperature with time as the input within the validity regime.

Generalizing the analysis, the variation of a dependent parameter in terms of independent parameter(s) (there could be multiple independent parameters) can be represented in terms of functional relationships as;

(a) $\eta = f(x)$

(b) $\eta = f(x, y)$

(c) $\eta = f(x, y, z)$

(d) $\eta = f(x, y, z, ...)$

In the described functional relations, the first corresponds to one dependent variable function, the second corresponds to two dependent variables function, the third corresponds to three dependent variables function and the last is the most general form with multiple dependent variables. The functional relation(s) can be established based on the knowledge of variation of dependent parameter with the independent parameter(s) and can be used to evaluate the magnitude of the dependent parameter for any particular set of independent parameter(s) within the validity regime.

With the knowledge of the functional relationship, the derivatives of the functional relationship can also be established; ordinary differential for single dependent variable function and partial differential for functions with two or more dependent variables. As an example, the first, second and third derivative of the function representing the variation of ambient temperature with time is represented as below;

(a) $T = -0.00001t^6 + 0.0009t^5 - 0.0233t^4 + 0.2667t^3 - 1.0472t^2 + 0.2637t + 23.8840$

(b) $\dfrac{dT}{dt} = -0.00006t^5 + 0.0045t^4 - 0.0932t^3 + 0.8001t^2 - 2.0944t + 0.2637$

(c) $\dfrac{d^2T}{dt^2} = -0.00030t^4 + 0.0180t^3 - 0.2796t^2 + 1.6002t - 2.0944$

(d) $\dfrac{d^3T}{dt^3} = -0.00120t^3 + 0.0540t^2 - 0.5592t + 1.6002$

Using the functional relation and its derivatives, at particular value of t, corresponding magnitudes can be estimated. For example, at time $t = 10$, $t = 15$ and $t = 20$ the value of the ambient temperature variation function and its three derivatives will be;

Table 1: Estimated values for curve fit function and its derivatives at particular value(s) of temperature

	T	$\frac{dT}{dt}$	$\frac{d^2T}{dt^2}$	$\frac{d^3T}{dt^3}$
t=10	30.04	2.23	-0.31	-0.22
t=15	34.61	-0.56	-0.50	0.14
t=20	29.06	-1.02	0.17	-0.05

Essentially, once the functional relationship is established, derivatives can be obtained and magnitude of the dependent parameter and its variation with the independent parameter(s) can be obtained at desired values of the independent parameter(s).

Now, let us consider a slightly different scenario in terms of the temperature measurement experiment. Consider the availability of a highly advanced instrument (like a thermal camera) which can at any instance of time simultaneously measure the temperature and the variation of temperature with time. Let the instrument display all possible derivatives of temperature with time along with the temperature at a particular instance. It is important to note that the instrument measures and displays the MAGNITUDE. Continuing with the example of ambient temperature measurement, at time $t = 15$ (hours) the instrument would indicate the following values;

T	$\frac{dT}{dt}$	$\frac{d^2T}{dt^2}$	$\frac{d^3T}{dt^3}$	$\frac{d^4T}{dt^4}$	$\frac{d^5T}{dt^5}$	$\frac{d^6T}{dt^6}$
deg	$\frac{deg}{h}$	$\frac{deg}{h^2}$	$\frac{deg}{h^3}$	$\frac{deg}{h^4}$	$\frac{deg}{h^5}$	$\frac{deg}{h^6}$
34.60683	-0.56393	-0.49761	0.13885	0.05164	-0.02176	-0.00833

One fundamental limitation of the instrument is that it permits only one measurement and continuous measurement is not possible. What it basically means is that measurement of T, $\frac{dT}{dt}$, $\frac{d^2T}{dt^2}$, $\frac{d^3T}{dt^3}$ and so on would be possible for one particular value of time (say $t = 18$ hours) but not consecutively (like $t = 1$,$t = 2$, $t = 3$ ... etc). This set of experiments is significant from the perspective of Taylor's series.

The take away from the current section is the understanding / appreciation of the fact that through experimental / other means, a data set containing the magnitude of a dependent variable and its derivatives can be generated at a particular value of independent parameter(s). This understanding shall now be carried forward to describe the Taylor's series.

2.2 Understanding Taylor's series

Continuing with the previous experiment of measurement of temperature and temperature change rate(s) with time, if the values are measured/estimated/known at some time t_1 then Taylor's series permits mathematical estimation of the temperature at some other time t_2 using the values of temperature and its change rate at t_1. This is expressed as;

$$T_{(t=t_2)} = f\left\{T_{(t=t_1)}, \quad \left.\frac{dT}{dt}\right|_{(t=t_1)}, \quad \left.\frac{d^2T}{dt^2}\right|_{(t=t_1)}, \quad \left.\frac{d^3T}{dt^3}\right|_{(t=t_1)}, ..., \quad \left.\frac{d^nT}{dt^n}\right|_{(t=t_1)}, ...\right\} \tag{1}$$

Thus, basically, with the knowledge of temperature and its change rate at only one particular time, the temperature at any other time can be estimated. This is the fundamental beauty of Taylor's series. The exact nature of the Taylor's series in line with the example under consideration will be of the form;

$$T_{(t=t_2)} = T_{(t=t_1)} + \frac{(t_2-t_1)}{1!}\left.\frac{dT}{dt}\right|_{(t=t_1)} + \frac{(t_2-t_1)^2}{2!}\left.\frac{d^2T}{dt^2}\right|_{(t=t_1)} + \frac{(t_2-t_1)^3}{3!}\left.\frac{d^3T}{dt^3}\right|_{(t=t_1)} + ... \tag{2}$$

In a more generic form, considering that the temperature varies with time, the dependence can be described in the form of the functional relation $T = f(t)$. Substituting $T = f(t)$ in the equation 2;

$$f(t_2) = f(t_1) + \frac{(t_2 - t_1)}{1!}\frac{df(t_1)}{dt} + \frac{(t_2 - t_1)^2}{2!}\frac{d^2f(t_1)}{dt^2} + \frac{(t_2 - t_1)^3}{3!}\frac{d^3f(t_1)}{dt^3} + ...$$

The term $(t_2 - t_1)$ can be represented as Δt resulting in;

$$f(t_2) = f(t_1) + \frac{(\Delta t)}{1!}\frac{df(t_1)}{dt} + \frac{(\Delta t)^2}{2!}\frac{d^2f(t_1)}{dt^2} + \frac{(\Delta t)^3}{3!}\frac{d^3f(t_1)}{dt^3} + ...$$

This is also expressed as;

$$f(t_2) = f(t_1) + \frac{(\Delta t)}{1!}\frac{d}{dt}f(t_1) + \frac{(\Delta t)^2}{2!}\frac{d^2}{dt^2}f(t_1) + \frac{(\Delta t)^3}{3!}\frac{d^3}{dt^3}f(t_1) + ...$$

It is important to note that, the Taylor's series permits the estimation of only the dependent parameter (in this particular case temperature T) and not the derivatives.

2.3 Taylor's series : Formal definition

The Taylor's series is an infinite series that permits the evolution of the dependent parameter with the variation of the independent parameter using the knowledge of the dependent parameter and its derivatives at a particular value of the independent parameter. The Taylor's series in a single variable with x being the independent and y being the dependent variable is represented as;

$$y_{(x=x_2)} = y_{(x=x_1)} + \frac{(x_2 - x_1)}{1!}\left.\frac{dy}{dx}\right|_{(x=x_1)} + \frac{(x_2 - x_1)^2}{2!}\left.\frac{d^2y}{dx^2}\right|_{(x=x_1)} + \frac{(x_2 - x_1)^3}{3!}\left.\frac{d^3y}{dx^3}\right|_{(x=x_1)} + ...$$

Representing $y = f(x)$ and $(x_2 - x_1) = h$

$$f(x_2) = f(x_1) + \frac{h}{1!}\frac{d}{dx}f(x_1) + \frac{h^2}{2!}\frac{d^2}{dx^2}f(x_1) + \frac{h^3}{3!}\frac{d^3}{dx^3}f(x_1) + ... \tag{3}$$

The above relation can be generalized further by considering the knowledge of dependent parameters and its derivatives at some generic value of the independent variable x and the Taylor's series is used to estimate the value of the dependent parameter some other position of $x + h$. The Taylor's series will then take the form;

$$f(x+h) = f(x) + \frac{h}{1!}\frac{d}{dx}f(x) + \frac{h^2}{2!}\frac{d^2}{dx^2}f(x) + \frac{h^3}{3!}\frac{d^3}{dx^3}f(x) + ... \tag{4}$$

For simplicity, the differentials are represented as $\frac{d}{dx}f(x) = f^{'}(x), \frac{d^2}{d^2x}f(x) = f^{''}(x)$ etc. Substituting in equation 4, the new simplified form will be;

$$f(x+h) = f(x) + \frac{h}{1!}f^{'}(x) + \frac{h^2}{2!}f^{''}(x) + \frac{h^3}{3!}f^{'''}(x) + ... \tag{5}$$

Equation 4 and/or 5 represent the most generic form of Taylor's series in a single variable. Replacing $x = x_1$ (the point where values of dependent parameter and its deliverables are available) and $h = x_2 - x_1$ in equation 4 or 5 results in the specific form of Taylor's series;

$$f(x_2) = f(x_1) + \frac{(x_2 - x_1)}{1!}f^{'}(x_1) + \frac{(x_2 - x_1)^2}{2!}f^{''}(x_1) + \frac{(x_2 - x_1)^3}{3!}f^{'''}(x_1) + ...$$

At this juncture, it emphasized that the Taylor's series DOES NOT require OR depend on the actual functional relation between the dependent and independent parameter. As a matter of fact, if the exact nature of the functional relationship ($y = f(x)$) were known then there would be no need to apply the Taylor's series at all.

2.4 Moving backwards

Having presented the formal definition of Taylor's series, it is important to note that Taylor's hypothesis is not restricted to functional assessment at higher values of independent parameter $(x + h)$ only. Functional assessment is possible even for lower values of independent parameter i.e., for $(x - h)$. Replacing h by $-h$ in equation 5 is all that is required for the assessment. Substituting h by $-h$ in equation 5 results in equation 6.

$$f(x-h) = f(x) - \frac{h}{1!}f^{'}(x) + \frac{h^2}{2!}f^{''}(x) - \frac{h^3}{3!}f^{'''}(x) + ... \tag{6}$$

2.5 Testing the hypothesis

The hypothesis of Taylor's series shall be put to test in the current section. The testing is performed by first defining a general function in a single variable $f(x)$. The values of the function and all its (possible) derivatives are estimated at a particular value of x, say x_i. Using these values in Taylor's series, the value of the functions shall be estimated at x_{i+1} (basically $x + h$). The value is also estimated at x_{i+1} using the function $f(x)$ i.e., $f(x_{i+1})$. The value at

x_{i+1} estimated using the Taylor's series is compared with the value estimated using the function $f(x)$ i.e., $f(x_{i+1})$ and if the two values match, the hypothesis is valid.

Consider the function as in equation 7. This function shall now become the base function for testing the Taylor's series Hypothesis.

$$f(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 \tag{7}$$

Prior to proceeding further, all possible derivatives of the function as in equation 7 shall be evaluated. It may be noted that since the function is a simple polynomial of sixth order, derivatives from the seventh order and higher will be zero. The differentiated function set will be as described below;

$$\frac{d}{dx}f(x) = 1 + 4x + 9x^2 + 16x^3 + 25x^4 + 36x^5$$
$$\frac{d^2}{dx^2}f(x) = 4 + 18x + 48x^2 + 100x^3 + 180x^4$$
$$\frac{d^3}{dx^3}f(x) = 18 + 96x + 300x^2 + 720x^3$$
$$\frac{d^4}{dx^4}f(x) = 96 + 600x + 2160x^2$$
$$\frac{d^5}{dx^5}f(x) = 600 + 4320x$$
$$\frac{d^6}{dx^6}f(x) = 4320$$
$$\frac{d^7}{dx^7}f(x) = 0 \quad ; \quad \text{all subsequent higher derivatives will also be zero}$$

To test the hypothesis, the value of the function and its derivatives shall be estimated at $x = 5$ and using these values in Taylor's series, the value function at $x = 8$ shall be estimated. The value of the function and its non zero derivatives at $x = 5$ are consolidated in the table below;

$f(5)$	$\frac{d}{dx}f(5)$	$\frac{d^2}{dx^2}f(5)$	$\frac{d^3}{dx^3}f(5)$	$\frac{d^4}{dx^4}f(5)$	$\frac{d^5}{dx^5}f(5)$	$\frac{d^6}{dx^6}f(5)$
112305	130371	126294	97998	57096	22200	4320

With the knowledge of the function and its derivatives at $x = 5$, the value of the function at $x = 8$ is estimated considering $8 = 5 + 3$ i.e., substituting $x = 5$ and $h = 3$ in equation 4, reproduced below for convenience;

$$f(x+h) = f(x) + \frac{h}{1!}\frac{d}{dx}f(x) + \frac{h^2}{2!}\frac{d^2}{dx^2}f(x) + \frac{h^3}{3!}\frac{d^3}{dx^3}f(x) + ...$$

Substituting the values in the above equation ($h = 3$ and other values from the above table), $f(8)$ evaluates to 1754760. Substituting $x = 8$ in equation 7 also yields $f(8)$ of 1754760, establishing the hypothesis of Taylor's series.

3 Taylor's series in multiple variables

The Taylor's series is not restricted to a single independent variable functional relation. It is equally applicable when there are multiple independent variables with the same underlying philosophy. One important difference that emerges in respect of multivariate Taylor's series is the introduction of partial differentials while the single variable Taylor's series has ordinary differential equations. The generic form of Taylor's series for one, two and three variables are of the form;

3.1 Single independent parameter

$$f(x+h) = f(x) + \frac{h}{1!}\frac{d}{dx}f(x) + \frac{h^2}{2!}\frac{d^2}{dx^2}f(x) + \frac{h^3}{3!}\frac{d^3}{dx^3}f(x) + ... \tag{8}$$

3.2 Two independent parameters

In this, some parameter is dependent on two independent parameters x and y and has a hypothetical functional relation $f(x, y)$. In its most generic form, with the knowledge of the dependent parameter and its derivatives at the value of the independent parameter pair $(x,\, y)$, the value of the dependent parameter at some other value of the independent pair represented by $(x+h,\, y+k)$ is represented as;

$$\begin{aligned} f(x+h, y+k) = f(x, y)+ & \\ & \frac{1}{1!}\left(h\frac{\delta}{\delta x}f(x,y) + k\frac{\delta}{\delta y}f(x,y)\right) + \\ & \frac{1}{2!}\left(h^2\frac{\delta^2}{\delta x^2}f(x,y) + k^2\frac{\delta^2}{\delta y^2}f(x,y) + 2hk\frac{\delta^2}{\delta xy}f(x,y)\right) + ... \end{aligned} \tag{9}$$

The use of partial derivatives is evident. For simplicity, the following modified approach to representing derivatives shall be used.

$$\frac{\delta}{\delta x}f(x,y) = f_x(x,y); \frac{\delta}{\delta y}f(x,y) = f_y(x,y); \frac{\delta^2}{\delta x^2}f(x,y) = f_{xx}(x,y); \frac{\delta^2}{\delta y^2}f(x,y) = f_{yy}(x,y)$$

$$\frac{\delta^2}{\delta xy}f(x,y) = f_{xy}(x,y)$$

$$\frac{\delta^3}{\delta x^2 \delta y}f(x,y) = f_{xxy}(x,y) = f_{x^2y}(x,y); \frac{\delta^3}{\delta x \delta y^2}f(x,y) = f_{xyy}(x,y) = f_{xy^2}(x,y)$$

The general pattern of representing the derivatives will be;

$$\frac{\delta^q}{\delta x^m \delta y^n}f(x,y) = f_{x^m y^n}(x,y) \qquad \text{It is important to note that } \; q = m + n$$

Adopting the modified derivative representation format, the Taylor's series in two variables can be represented as;

$$
\begin{aligned}
f(x+h, y+k) = f(x,y) + &\\
&\frac{1}{1!}\left[hf_x(x,y) + kf_y(x,y)\right] + \\
&\frac{1}{2!}\left[h^2 f_{x^2}(x,y) + k^2 f_{y^2}(x,y) + 2hkf_{xy}(x,y)\right] + \\
&\frac{1}{3!}\left[h^3 f_{x^3}(x,y) + k^3 f_{y^3}(x,y) + 3h^2 k f_{x^2y}(x,y) + 3hk^2 f_{xy^2}(x,y)\right] + ...
\end{aligned}
\tag{10}
$$

3.3 Three independent parameters

Continuing with the methodology adopted for the two independent variable system, the final form of the three variable system with x, y and z being the independent parameters will be as below. For simplicity $f(x,y,z)$ will be written only as f.

$$
\begin{aligned}
f(x+h, y+k, z+l) = f + &\\
&\frac{1}{1!}\left[hf_x + kf_y + lf_z\right] + \\
&\frac{1}{2!}\left[h^2 f_{x^2} + k^2 f_{y^2} + l^2 f_{z^2} + 2hkf_{xy} + 2hlf_{xz} + 2klf_{yz}\right] + ...
\end{aligned}
\tag{11}
$$

4 Pattern for evolving Taylor's series

Reviewing the single, double and triple variable Taylor's series, the pattern for extension of multi-variable Taylor's series into infinite series is generally evident. The general rule for the extension of Taylor's series is formally presented as below. The general rule is built up by describing the rule for single, double and triple variable Taylor's series individually.

4.1 Consideration of symbols for pattern evolution

Towards simplifying the approach to evolving a generic pattern, the following representation methods are considered. In the representation, h, k, l represent the displacement in the x, y, z directions.

$$
a^1 = hf_x = h\frac{df}{dx}\ ;\ a^2 = h^2 f_{x^2} = h^2\frac{d^2 f}{dx^2}\ ;\ a^3 = h^3 f_{x^3} = h^3\frac{d^3 f}{dx^3}\ ;.....\ a^n = h^n f_{x^n} = h^n\frac{d^n f}{dx^n}
$$

$$
b^1 = kf_y = k\frac{df}{dy}\ ;\ b^2 = k^2 f_{y^2} = k^2\frac{d^2 f}{dy^2}\ ;\ b^3 = k^3 f_{y^3} = k^3\frac{d^3 f}{dy^3}\ ;.....\ b^n = k^n f_{y^n} = k^n\frac{d^n f}{dy^n}
$$

$$c^1 = lf_z = l\frac{df}{dz}\ ;\ c^2 = l^2 f_{z^2} = l^2\frac{d^2 f}{dz^2}\ ;\ c^3 = l^3 f_{z^3} = l^3\frac{d^3 f}{dz^3}\ ;\ .....\ c^n = l^n f_{z^n} = l^n\frac{d^n f}{dz^n}$$

The pattern can be extended to any number of variables. Further, for multiple variables, partial derivatives are involved following the pattern indicated as below, a logical extension of the aforementioned pattern;

$$ab = hk\frac{\delta^2 f}{\delta x \delta y}\ ;\ a^2 b = h^2 k\frac{\delta^2 f}{\delta x^2 \delta y}\ ;\ ab^2 = hk^2\frac{\delta^2 f}{\delta x \delta y^2}\ ;\ .....\ a^m b^n = h^m k^n\frac{\delta^{m+n} f}{\delta x^m \delta y^n}$$

In general;

$$a^m b^n c^o = h^m k^n l^o \frac{\delta^{m+n+o} f}{\delta x^m \delta y^n \delta l^o}$$

The described general approach can be extended to any number of variables. The defined symbols a^m; b^n; c^o ..... and their combinations are now used in the generalizing the Taylor's series expansion.

4.2 Single variable Taylor's series generalization

$$f(x+h) = f + \frac{1}{1!}hf_x + \frac{1}{2!}h^2 f_{x^2} + \frac{1}{3!}h^3 f_{x^3} + ...\frac{1}{n!}h^n f_{x^n} + ...$$

$$f(x+h) = f + \sum_{n=1}^{n=\infty} \frac{1}{n!} h^n f_{x^n}$$

$$f(x+h) = f + \sum_{n=1}^{n=\infty} \frac{1}{n!} a^n \qquad \text{where} \qquad a^n = h^n f_{x^n} = h^n \frac{d^n f}{dx^n}$$

For functional assessment at lower values of independent variable i.e., for assessing $f(x-h)$, h should be replaced by $-h$.

4.3 Two variable Taylor's series generalization

$$f(x+h, y+k) = f + \frac{1}{1!}\left[hf_x + kf_y\right] + \frac{1}{2!}\left[h^2 f_{x^2} + k^2 f_{y^2} + 2hkf_{xy}\right] + \frac{1}{3!}\left[h^3 f_{x^3} + k^3 f_{y^3} + 3h^2 k f_{x^2 y} + 3hk^2 f_{xy^2}\right] + ...$$

$$f(x+h, y+k) = f + \sum_{n=1}^{n=\infty} \frac{1}{n!}(a+b)^n$$

For functional assessment at lower values of independent variables i.e., for assessing $f(x-h, y-k)$, h and k should be replaced by $-h$ and $-k$ respectively. For functional assessment of $f(x-h, y+k)$, only h is replaced by $-h$ in the general equation. For the case of $f(x+h, y-k)$, only k is replaced by $-k$ in the general equation.

4.4 Three variable Taylor's series generalization

$$f(x+h, y+k, z+l) = f + \frac{1}{1!}\left[hf_x + kf_y + lf_z\right] + \frac{1}{2!}\left[h^2 f_{x^2} + k^2 f_{y^2} + l^2 f_{z^2} + 2hkf_{xy} + 2hlf_{xz} + 2klf_{yz}\right] + ...$$

$$f(x+h, y+k, z+l) = f + \sum_{n=1}^{n=\infty} \frac{1}{n!}(a+b+c)^n$$

For functional assessment at lower values of independent variables, the same logic as described in the single and two variable generalisation sub-sections shall be applicable.

4.5 'n' variable Taylor's series generalization

$$f(x+h, y+k, z+l, ...) = f + \sum_{n=1}^{n=\infty} \frac{1}{n!}(a+b+c+...)^n$$

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